Statistical Error Calibration for Hatch-filtered Ranges using Spectral Characterization of Raw Measurement Errors

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BIOGRAPHY

Johannes Mach completed a master course in Geodesy at the University of Technology in Graz, Austria. After his graduation in 2001 he started working in the field of applied satellite navigation. It was during a young graduate traineeship at ESA's Galileo Project Office that he took a special interest in GNSS system design and performance aspects. Since 2004 he has been working as systems engineer in the integrity systems department of IFEN GmbH, where he is concerned with algorithm design, software development and performance analysis for several integrity related projects.

Ingrid Deuster holds a diploma in Geodesy from the University of Stuttgart. She joined IFEN in 1999 as a systems engineer and has been working since then in the field of GNSS integrity processing. Lately she has been dealing with integrity performance experimentation in the GSTB-V1 project and the development of IFEN's Experimental IPF for Galileo.

Robert Wolf holds a degree in Aerospace Engineering from the Technical University of Munich and a Ph. D. from University FAF Munich. He joined IFEN GmbH in 1999. Since 2001 he is company responsible for all integrity related projects. Among other topics he is currently involved in the development of the Processing Facility of the German Galileo Test Bed (GATE).

ABSTRACT

The presented error calibration concept for Hatch-filtered ranges makes use of a recursive error propagation algorithm, which is based on the covariance analysis for a Kalman filter with sub-optimal gain filters. This approach naturally covers all possible geometries and line-of-sight histories, because the error propagator can always be updated in parallel with the actual Hatch filter.

Under the condition that the underlying stochastic model is representative for the true error behavior, the error propagation algorithm produces optimal a-priori error estimates, also before filter convergence.

In addition a suitable calibration method has been developed that allows to determine the necessary stochastic model parameters as a function of elevation. The calibration includes a simple but efficient characterization of spectral properties. It is based on raw pseudo-range data rather than filtered ranges. This is another important advantage, because the calibration does not have to be repeated in case of change of algorithm parameters (e.g. filter constant).

Due to these advantages the concept is considered an interesting alternative to the usual direct calibration of smoothed range errors. Possible fields of application are user positioning as well as system monitoring algorithms.

INTRODUCTION

The use of Hatch filtering is a basic pre-processing method, which is widely used in both user positioning and system monitoring algorithms. It allows smoothing noisy pseudo-range observations with the aid of the by far more precise carrier phase measurements. Thereby Hatch filtering represents a convenient method to take benefit of the good carrier phase accuracy even without ambiguity resolution. For example it is part of the standard positioning algorithm for civil aviation, cf. RTCA (2001).

For many applications it is also desirable to have representative a-priori error measures for these Hatch-filtered pseudo-ranges, which are typically the input data to least-squares estimation. Such error measures, mostly in terms of standard deviation, play a particularly important role in safety-critical systems, e.g. the Galileo integrity monitoring.

However, the provision of optimal error measures is not so obvious, because the error of the Hatch filter output is a function of the errors of all input observations since filter initialization. Whereas the error of raw pseudo-ranges can usually be calibrated as an elevation-dependent function, the error of a smoothed range depends on the entire history of the given line-of-sight, including previous elevation angles, the filter time (time elapsed since filter initialization) and data gaps.

This characteristic property of the Hatch filter makes "direct calibration" of smoothed ranges quite difficult, because in principle statistical calibration tables have to be established with respect to both elevation and filter time. However this requires an even larger amount of archived data and the results are still not accurate, because this approach does not account for data gaps and the exact elevation history.

Therefore calibration is usually performed only with respect to elevation and only for the smoothed range error after filter convergence. This approach is feasible but clearly not optimal, since lines-of-sight can be used only after filter convergence and the elevation dependence has to be conservative to cover all situations. For example, the calibration for a low elevation has to cover the case of an ascending satellite (even lower elevations in the past), whereas the accuracy of a descending line-of-sight (higher elevations in the past) might be significantly better.

The algorithm proposed in this paper is able to overcome the mentioned drawbacks by means of individual covariance propagation for each line-of-sight, always in parallel with the corresponding smoothing filter. Unlike other concepts it is based on statistical calibration of raw (unfiltered) pseudo-range errors, including its spectral characterization in terms of auto-correlation.

The following sections describe the new error propagation algorithm, a possible error model (still to be validated in more detail) and a suitable calibration method.

THE COLOURED NOISE PROBLEM

The standard Hatch filter is defined by the following recursive formula (cf. Hofmann-Wellenhof et al., 1998):

$$\hat{R}_k = (1-w_k)(\hat{R}_{k-1} + \varphi_k - \varphi_{k-1}) + w_k R_k$$

$$w_k = \begin{cases} 1/k & \text{for } k < T_{flt} \\ 1/T_{flt} & \text{for } k \geq T_{flt} \end{cases}$$

with

$\hat{R}_i$ .... smoothed pseudo-range (epoch i)

R_i .... raw pseudo-range (epoch i)

φ_i .... carrier phase (epoch i)

w_i .... weight factor (epoch i)

T_{flt} .... filter constant (e.g. 300 seconds)

The above formula shows that at any epoch the smoothed range is a linear combination of the smoothed range of the previous epoch as well as "raw" pseudo-range and carrier phase observations. Therefore it appears possible to use simple standard covariance propagation for linear functions to propagate the variance. This would indeed be possible for "white" noise errors. However, pseudo-range errors usually are affected by time-correlated "coloured" noise, which is mainly caused by multi-path effects. As a consequence the new observation in general is correlated with the smoothed range of the previous epoch. This correlation somehow has to be taken into account, otherwise results will be wrong (usually optimistic).

One option would be to use the explicit (non-recursive) expression of the smoothed range as linear function of raw observations only.

$$\hat{R}_k = \sum_{i=0}^{k-1} \overline{w}_{k,i} (R_{k-i} + \varphi_k - \varphi_{k-i})$$

with $\quad \overline{w}_{k,i} = w_{k-i} \prod_{j=0}^{i-1} (1 - w_{k-j})$

In that case one has only to deal with correlations between raw measurements, which can be derived from a model. Unfortunately the Hatch filter has an infinite impulse response, which means that the above sum can become very long (theoretically infinite), at least if exact computation is desired (without truncation). Due to the resulting large matrix multiplications this approach is very inconvenient and rather unsuitable for practical processing with many lines-of-sight.

Therefore we are considering a recursive algorithm, which is presented in the following section. As explained above, for a recursive algorithm it is always necessary to model the correlation between raw observations and smoothed ones, and this is what represents the main difficulty in this context.

ERROR PROPAGATION ALGORITHM

Kalman-like formulation of Hatch filter

The Hatch filter output is computed as a weighted average of the smoothed pseudo-range of the previous epoch, propagated to the current epoch using phase observations, and the current raw pseudo-range. This is equivalent to a Kalman filter recursion, with the important difference that the weight of the new observation is pre-determined and not computed from a stochastic model. In fact, the following matrix notation of the Hatch filter

$$\begin{bmatrix}\hat{R}_k \\ \hat{\varphi}_k\end{bmatrix} = \begin{bmatrix}\hat{R}_{k-1} \\ \hat{\varphi}_{k-1}\end{bmatrix} + \begin{bmatrix}w_k & 1-w_k \\ 0 & 1\end{bmatrix}\left(\begin{bmatrix}R_k \\ \varphi_k\end{bmatrix} - \begin{bmatrix}1 & 0 \\ 0 & 1\end{bmatrix}\begin{bmatrix}\hat{R}_{k-1} \\ \hat{\varphi}_{k-1}\end{bmatrix}\right)$$

exactly corresponds to the basic equations of a Kalman filter:

Observation update: $\hat{x}_k = \hat{x}_k^{pred} + K_k\left(z_k - H_k \hat{x}_k^{pred}\right)$

Time update: $\hat{x}_k^{pred} = \Phi_k \hat{x}_{k-1}$

The corresponding gain (K), observation (H) and transition (Φ) matrices can be easily formed by comparison of the above recursions.

Simplified formulation

Alternatively one can also use a simplified Kalman-like formulation of the Hatch filter with a one-element state vector representing the smoothed range.

Observation update: $\hat{R}_k = \hat{R}_k^{pred} + w_k\left(R_k - \hat{R}_k^{pred}\right)$

Time update: $\hat{R}_k^{pred} = \hat{R}_{k-1} + \left(\varphi_k - \varphi_{k-1}\right)$

The carrier phase is regarded here as deterministic parameter and is used only for prediction of the next epoch. This formulation of course neglects the carrier phase error, which indeed can be regarded as minor with respect to the pseudo-range error.

State-vector augmentation

The above models (exact and simplified) still do not allow covariance propagation in case of colored observation noise, because the Kalman formulas are based on the assumption that the observation vector and the state vector are uncorrelated. As already explained in the previous section, this is generally not the case for the Hatch filter in presence of time-correlated pseudo-range errors.

Fortunately this problem can be overcome by the well-known technique of state-vector augmentation. The time-correlated part of the observation error (ε_{CN}) is simply shifted from the observation to the parameter side of the observation equation, where it is modeled by means of one or several additional state-vector parameters.

$$\begin{aligned} R &= \varepsilon_{CN+WN} \\ R + c_{CN} &= \varepsilon_{WN} \end{aligned}$$

Thereby only the white noise part (ε_{WN}) remains on the right hand side of the equation, which is obviously not correlated with any of the state vector parameters.

Note that the algorithm can easily be adapted to any stochastic model. For the present study it was decided to consider a 1st order Gauss-Markov (GM) process, which can be modeled with only one additional state-vector element. The GM process is completely defined by two parameters

σ_{GM} .... standard deviation of GM process

T_{flt} .... correlation time of GM process

and is generated from a Gaussian white noise process with unit standard deviation (denoted by n_i) by the following recursion:

$$\begin{aligned} x_0 &= \sigma_{GM} n_0 \\ x_k &= e^{-\beta(t_k - t_{k-1})} x_{k-1} + \sigma_{GM}\left(1 - e^{-\beta(t_k - t_{k-1})}\right) n_k \end{aligned}$$

The parameter β is the reciprocal of the correlation time.

$$\beta = 1/T_{flt}$$

It is convenient to use a unit GM process (with standard deviation one) in the state-vector and to scale it to the needed standard deviation, which can change with respect to time and in this context typically is a function of the elevation angle. Thus the observation equation takes on the following form:

$$R + \sigma_{GM}(\varepsilon) x_{GM,unit} = \varepsilon_{WN}$$

The observation model is equivalently expressed by the observation matrix (H) and observation covariance matrix (R):

$$H_k = \begin{bmatrix}1 & \sigma_{GM}\end{bmatrix} \qquad R_k = \begin{bmatrix}\sigma_{WN}^2\end{bmatrix}$$

The dynamic and stochastic models are defined by the transition matrix (Φ) and process noise matrix (Q):

$$\Phi_k = \begin{bmatrix}1 & 0 \\ 0 & e^{-\beta(t_k - t_{k-1})}\end{bmatrix}$$

$$Q_k = \begin{bmatrix} 0 & 0 \\ 0 & 1 - e^{-2\beta(t_k - t_{k-1})} \end{bmatrix}$$

The decisive point is that not the optimal Kalman gain matrix is used, but instead the gain factor of the range parameter is set to the weight used by the Hatch filter and the gain factor associated with the GM parameter is set to zero.

$$K_k = \begin{bmatrix} w_k \\ 0 \end{bmatrix}$$

This means that the GM parameter is a pure nuisance parameter, which is not estimated. Nevertheless it influences the state vector covariance matrix through the observation equation.

The observation update of the covariance matrix can be performed by means of the Kalman filter equation for "sub-optimal" gain factors (cf. Brown, 1997):

$$P_k = (I - K_k H_k) P_k^{pred} (I - K_k H_k)^T + K_k R_k K_k^T$$

Finally, the time update is computed using the familiar propagation formula:

$$P_k^{pred} = \Phi_k P_{k-1} \Phi_k^T + Q_k$$

This completes the necessary set of formulas. The recursion is performed in parallel to the Hatch filter. Whenever a pseudo-range is fed into the Hatch filter, the covariance matrix is first propagated to the current epoch (time update) and subsequently updated with the same gain factor as the Hatch filter (observation update). In this way the covariance matrix always represents the current state of the Hatch filter.

STOCHASTIC MODEL

This section briefly describes a simple but effective stochastic error model, which is proposed for usage in combination with the algorithm presented above. It is a three-component model consisting of:

- A white noise component (characterized by σ_{WN})
- A Gauss-Markov component (σ_{GM}, T_{flt})
- An additional noise floor (σ_{NF})

While the white noise refers to the errors due to receiver noise and interference, the colored noise (GM) refers to slowly varying errors e.g. caused by multipath. The "noise floor" component stands for an additional error, which is constant or has so strong time-correlation that it cannot be eliminated by any filtering.

The noise floor parameter looks somewhat artificial but is nevertheless useful. It represents a lower bound for the error of the smoothed ranges. Furthermore it can also be used to account for the carrier phase error, which is neglected by the simplified error propagation algorithm.

The resulting auto-correlation function of the pseudo-range error is shown in the figure below.

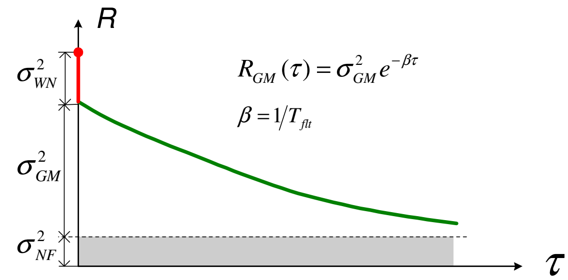

Figure 1: Auto-correlation function for 3-component model

The model is fully characterized by only 4 parameters, which are assumed to be a function of elevation. The necessary calibration of the model parameters is described in the next section.

CALIBRATION OF MODEL PARAMETERS

In principle all model parameters can be determined directly from the auto-correlation function, which represents the spectral behavior of the error (equivalently to the spectral power density function). For this purpose the auto-correlation function has to be empirically sampled for a suitable set of time differences τ_i.

This can be done either by using the conventional auto-correlation formula

$$R(\tau) = \tfrac{1}{N} \sum_{i=0}^{N-1} \left[y(t_i)\, y(t_i + \tau) \right]$$

or alternatively via a "de-correlation" parameter, which can be estimated using a time difference operator.

$$D(\tau) = \tfrac{1}{2N} \sum_{i=0}^{N-1} \left[y(t_i) - y(t_i + \tau) \right]^2$$
$$R(\tau) = \sigma^2 - D(\tau)$$

The theoretical equivalence of the two methods can be easily shown by the following derivation:

$$D(\tau) = \tfrac{1}{2} E\left\{ \left[y(t_i) - y(t_i+\tau) \right]^2 \right\}$$
$$= \tfrac{1}{2} E\left\{ y(t_i)^2 - 2y(t_i)y(t_i+\tau) + y(t_i+\tau)^2 \right\}$$
$$= E\left\{ y(t_i)^2 \right\} - E\left\{ y(t_i)y(t_i+\tau) \right\}$$
$$= \sigma^2 - R(\tau)$$

The advantage of the de-correlation is that for strongly correlated input signals it converges more quickly than the auto-correlation, because the time differences are much less correlated than the values themselves. On the other hand the calibration of the total variance ($\tau = 0$) remains the same and still needs a lot of data in case of high auto-correlation.

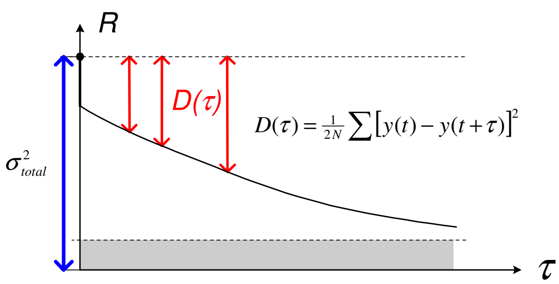

Figure 2: Reconstruction of auto-correlation function using statistical de-correlation

What remains to be done is to fit the white noise and Gauss-Markov parameters to the statistically reconstructed auto-correlation curve. It is rather hard to estimate the noise floor value from the auto-correlation function, which has to be chosen based on analysis or assumptions. It should be at least as large as the phase noise standard deviation.

To do the curve fitting the statistical auto-correlation is required only for a limited set of τ-values. (This is the main reason, why it is more efficient to work with the auto-correlation than with the power spectrum.) A possible choice of τ-values is

$$\tau = \{ 0, 1, 2, 4, 8, 16, \ldots \}$$

It reflects the fact that the model auto-correlation changes more slowly for large τ-values than for small ones.

An approximate estimate of the GM parameters can already be obtained by fitting a straight line to the auto-correlation values of just a few low τ-values. The approximate variance can be retrieved from the intersection of the fitted line with the y-axis, the correlation time from the inclination (see Figure 3). More exact values can be found by means of iterative estimation of the shape parameters of an exponential function.

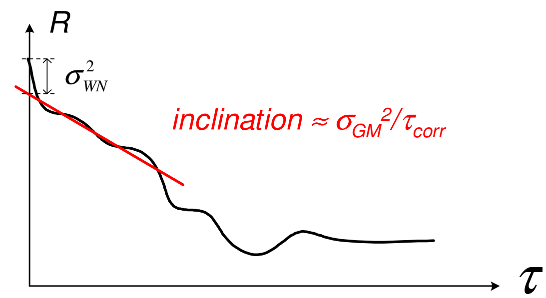

Figure 3: Approximate determination of GM parameters by means of line fitting

The calibration can be performed e.g. based on the code-minus-carrier (CmC) indicator. The CmC must be corrected for its arbitrary mean value, which can be determined for each uninterrupted tracking period of a given line-of-sight. Furthermore the CmC should be computed from the ionospheric-free code and phase combinations, as these are not affected by code-carrier divergence due to ionospheric refraction.

Since the stochastic properties of pseudo-range errors are assumed to be a function of the elevation angles, the auto-correlation function has to be determined and analyzed for a set of elevation classes (bins). For the application in the error propagation algorithm the parameters can then be linearly interpolated between elevation bins.

The following list summarizes the steps that are necessary for the calibration of e.g. a monitoring receiver:

- Compute the ionospheric-free code and phase combination for a sufficiently large period of time (several days)
- Compute the code-minus-carrier combination and correct it for the mean value per uninterrupted tracking period
- Estimate statistically the total error variance (per elevation bin)
- Estimate statistically the de-correlation parameter for a suitable set of τ-values (per elevation bin)
- Reconstruct the auto-correlation function based on total variance and de-correlation estimates (per elevation bin)
- Define a reasonable noise floor variance (σ^2_{NF})
- Estimate the Gauss-Markov parameters (σ^2_{GM}, β) using curve fitting and taking into account the noise floor variance (per elevation bin)

- Compute for each elevation bin the white noise variance (σ^2_{WN}) according to the below formula

$$\sigma^2_{WN} = \sigma^2_{total} - \sigma^2_{GM} - \sigma^2_{NF}$$

SIMULATION EXAMPLES

The following two simple examples show that the error propagation algorithm yields valid results.

Pure white noise case

Input error: $\sigma_{WN} = 1,\ \sigma_{GM} = \sigma_{NF} = 0$

Hatch filter: $T_{flt} = 100$

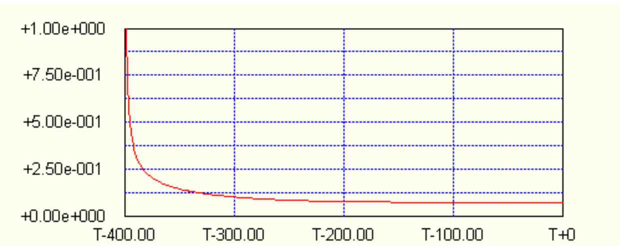

Figure 4: Time series of standard deviation of smoothed ranges (pure white noise case)

As shown by Figure 4 the computed standard deviation of the smoothed ranges indeed approaches the theoretical limit value (0.071) defined by the following equation:

$$\sigma^2_{smo} = \frac{\sigma^2_{raw}}{2T_{flt}}$$

The plot further shows that the best smoothing performance is reached only after a period equal to 2-3 times the filter constant.

Pure Gauss-Markov case

Input error: $\sigma_{GM} = 1,\ \tau_{corr} = 50$ s

Hatch filter: $T_{flt} = 100$

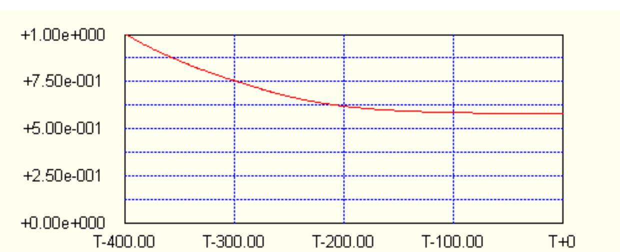

Figure 5: Time series of standard deviation of smoothed ranges (pure Gauss-Markov case)

The resulting smoothing error standard deviation after filter convergence (see Figure 5) is again consistent with the theoretical limit value (0.577), which can be calculated using the following relation (based on Brown, 1997):

$$\sigma^2_{smo} = \sigma^2_{raw} \frac{\tau}{\tau + T_{flt}}$$

The comparison of the two examples demonstrates very well the enormous impact of colored noise on the accuracy improvement that can be reached by Hatch filtering.

CONCLUSIONS

Summarizing, a complete concept for the error calibration of Hatch-filtered ranges has been developed, which is based on a recursive variance propagation algorithm and a suitable method for the calibration of the stochastic model parameters.

The new approach is regarded as a very promising option for usage in user positioning algorithms and (integrity) monitoring systems.

Test computations, which have been carried out with a software prototype, confirm the validity of the algorithm and the correctness of the theoretical analysis.

The next step should be a detailed assessment of the adequacy of the proposed 3-component stochastic model.

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